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VARIATIONAL PRINCIPLES IN THE MATHEMATICAL

THEORY OF PLASTICITY

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VARIATIONAL PRINCIPLES IN THE MATHEMATICAL
THEORY OF PLASTICITY[*]

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Summary

The fundamental definitions of work-hardening and perfect plasticity have been shown to have strong implications with respect to uniqueness of solution for elastic-plastic bodies. It is not surprising, therefore, to find that they lead rather directly to the variational principles as well. Perfect plasticity theory and both the incrementally linear and the incrementally non-linear theories for work-hardening materials are considered. The several counterparts of the minimum potential energy and the minimum complementary energy theorems are derived in a unified manner for stress-strain relations of great generality. Absolute minimum principles rather than relative are established.

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Introduction

There are any number of approaches to the establishment of variational principles. One is to state the principles directly and then proceed to prove them. Although clear and precise statements can be made, the motivation for the original inspiration does not appear. A newcomer to the field then frequently will be unable to appreciate the development and generally will not see how to produce appropriate theorems or modifications of his own. The approach to be followed here does not suppose the result to be known in advance. It is synthetic in a sense because the basic theorems have been stated and proved for a number of special materials [1][2][5].[*] Nevertheless, it is a procedure which arises logically from fundamental postulates in elasticity and in plasticity theory, and it is systematic.

In the theory of elasticity, whether linear or non-linear, the steps are reasonably straightforward. The equation of virtual work is written first under the implicit assumption of continuity of displacement and what may be termed equilibrium continuity of the stresses (surface tractions must be continuous across any surface but the normal stress components parallel to the surface may be discontinuous). In a common notation, repeated subscripts indicating summation:

$$\int_A T_i^* u_i dA + \int_V F_i^* u_i dV = \int_V \sigma_{ij}^* \epsilon_{ij} dV \tag{1}$$

The starred quantities are related through equilibrium and the unstarred are compatible. There need be no relation between the two sets of quantities. For convenience, the surface area A is divided into the region A_T on which the surface tractions T_i are specified and the region A_u over

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[*]Numbers in square brackets refer to the bibliography at the end of the paper.

which displacements u_i are given. The true and unique solution (no buckling, no initial stress) to the boundary value problem with given body forces F_i thus satisfies

$$\int_V \sigma_{ij}^t \varepsilon_{ij}^t dV - \int_{A_u} T_i^t u_i dA - \int_{A_T} T_i u_i^t dA - \int_V F_i u_i^t dV = 0 \qquad (2)$$

If approximate solutions are sought, two procedures suggest themselves immediately. One is to choose a compatible strain-displacement field ε_{ij}^c, u_i^c and satisfy the boundary conditions on A_u. The other is to select an equilibrium stress field σ_{ij}^E which satisfies the surface traction boundary conditions on A_T. More elaborate mixed schemes may be devised but they cannot be classed as obvious [3].

The value of an approximation procedure, or of a guess, must be determined by comparison of the approximate solution with the unknown true answer. The real difficulty and the intuitive heart of the problem lies in the decision on what should be compared. As has been noted, the equation of virtual work will be satisfied if the natural strains and displacements are replaced by any chosen set satisfying compatibility and the boundary conditions on A_u. Therefore

$$\int_V \sigma_{ij}^t \varepsilon_{ij}^c dV - \int_{A_T} T_i u_i^c dA - \int_V F_i u_i^c dV = \int_V \sigma_{ij}^t \varepsilon_{ij}^t dV - \int_{A_T} T_i u_i^t dA - \int_V F_i u_i^t dV \quad (3)$$

Transposing and calling the difference between the true and the assumed solution $\Delta\varepsilon_{ij}$, Δu_i

$$\int_V \sigma_{ij}^t \Delta\varepsilon_{ij} dV - \int_{A_T} T_i \Delta u_i dA - \int_V F_i \Delta u_i dV = 0 \qquad (4)$$

This form suggests strongly a consideration of the elastic strain energy density written as a function of strain alone

$$W(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \qquad (5)$$

because $dW = \dfrac{\partial W}{\partial \varepsilon_{ij}} d\varepsilon_{ij} = \sigma_{ij} d\varepsilon_{ij}.$ Equation (4) can then be restated as

$$\delta_\varepsilon \left[\int_V W(\varepsilon_{ij}^{\,t}) dV - \int_{A_T} T_i u_i^t dA - \int_V F_i u_i^t dV \right] \equiv \delta\,[P.E.^t] = 0 \qquad (6)$$

where δ_ε is to be interpreted as the first variation of the expression
in brackets (the potential energy) as u_i and ε_{ij} are varied in
accordance with compatibility and the boundary conditions on u_i.

A variational principle is not necessarily very helpful in solving
problems. The assumed state may not be close to the true one. What is
required instead is an absolute maximum or minimum principle, a comparison
of the value of the potential energy for the assumed state with that of
the true state, without restriction on the magnitude of the difference
between the states. The presentation here is, however, within the frame-
work of small displacement theory. A comparison may be made with the aid
of the identity

$$\int_V W(\varepsilon_{ij}^{\,c}) dV - \int_{A_T} T_i u_i^c dA - \int_V F_i u_i^c dV = \int_V W(\varepsilon_{ij}^{\,t}) dV - \int_{A_T} T_i u_i^t dA - \int_V F_i u_i^t dV$$

$$+ \int_V [W(\varepsilon_{ij}^{\,c}) - W(\varepsilon_{ij}^{\,t})] dV - \int_{A_T} T_i \Delta u_i dA - \int_V F_i \Delta u_i dV \qquad (7)$$

In view of (4), therefore, the potential energy of any admissible compatible
state is algebraically more than the potential energy of the true state by

$$P.E.^c - P.E.^t = \int_V [\,W(\varepsilon_{ij}^{\,c}) - W(\varepsilon_{ij}^{\,t}) - \sigma_{ij}^{\,t} \Delta\varepsilon_{ij}\,] dV \qquad (8)$$

The integrand may be rewritten as

$$\int_0^{\varepsilon_{ij}^{\,c}} \sigma_{ij} d\varepsilon_{ij} - \int_0^{\varepsilon_{ij}^{\,t}} \sigma_{ij} d\varepsilon_{ij} - \sigma_{ij}^{\,t} \Delta\varepsilon_{ij} = \int_{\varepsilon_{ij}^{\,t}}^{\varepsilon_{ij}^{\,c}} (\sigma_{ij} - \sigma_{ij}^{\,t}) d\varepsilon_{ij} \qquad (9)$$

The rectangles in Fig. 1 symbolize $\sigma_{ij}^t \Delta\epsilon_{ij}$. The shaded triangles represent (9), the integrand of (8). In Figs. 1a, b, c the triangles are on the positive strain energy side of the symbolic stress-strain curves for any magnitude $\Delta\epsilon_{ij}$. The potential energy is an absolute minimum for a linear or for a stable non-linear elastic material. For an unstable material, Figs. 1d, e, the shaded triangles are on the negative side for some $\Delta\epsilon_{ij}$ and the potential energy is not an absolute minimum.

A similar set of steps leads to the principle of minimum complementary energy. Equation (2) is satisfied if the σ_{ij}^t, T_i, F_i system is replaced by any other in equilibrium. For any state of stress σ_{ij}^E which satisfies the boundary conditions on A_T and is in equilibrium with F_i.

$$\int_V \Delta\sigma_{ij}\epsilon_{ij}^t dV - \int_{A_u} \Delta T_i u_i dA = 0 \tag{10}$$

where $\sigma_{ij}^E - \sigma_{ij}^t = \Delta\sigma_{ij}$ and ΔT_i is the corresponding change in surface traction on A_u.

The complementary energy density as a function of stress alone is suggested by the first integral.

$$\Omega(\sigma_{ij}) = \int_0^{\sigma_{ij}} \epsilon_{ij} d\sigma_{ij} \tag{11}$$

because $d\Omega = \dfrac{\partial\Omega}{\partial\sigma_{ij}} d\sigma_{ij} = \epsilon_{ij} d\sigma_{ij}$. Equation (10) can then be restated as

$$\delta_\sigma\left[\int_V \Omega(\sigma_{ij}^t) dV - \int_{A_u} T_i^t u_i dA\right] \equiv \delta_\sigma[C.E.^t] = 0 \tag{12}$$

where δ_σ is to be interpreted as the first variation of the expression in brackets (complementary energy) as σ_{ij} and T_i are varied in accordance with equilibrium with F_i and the boundary conditions on A_T. Fig. 2

is symbolic of the fact that the complementary energy is an absolute minimum for a stable material. Corresponding to (8) and (9)

$$C.E.^{E} - C.E.^{t} = \int_{V}\left[\Omega(\sigma_{ij}^{E}) - \Omega(\sigma_{ij}^{t}) - \epsilon_{ij}^{t}\Delta\sigma_{ij}\right]dV \tag{13}$$

where the integrand may be rewritten as

$$\int_{\sigma_{ij}^{t}}^{\sigma_{ij}^{E}} (\epsilon_{ij} - \epsilon_{ij}^{t})d\sigma_{ij} \tag{14}$$

The symbolic representation of the stress and strain tensors by one dimension each in Figs. 1 and 2 and the terms stable and unstable can be given general meaning and made precise.

The Fundamental Postulate for Elasticity and Plasticity

A basic postulate has been formulated for both elastic and plastic media [4] without time effects. It is essentially a definition of a stable material and may be stated as follows:

No work can be extracted from the material and the system of

forces acting upon it.

A more useful statement is in terms of an external agency which applies a set of additional forces to the body under a given load and then removes the added forces. The external agency must do positive work in the application of force. Over the cycle of application and removal the work done by the external agency must be positive if plastic deformation occurs in work-hardening material and will be zero if elastic changes only take place. For a perfectly plastic material, the work done by the external agency may be zero also when plastic deformation takes place although generally it will be positive.

The basic postulate may be applied to a homogeneous material under homogeneous stress σ_{ij}^{a} and strain ϵ_{ij}^{a}. Suppose the external agency

changes the state of stress by $\Delta\sigma_{ij}$ to σ_{ij}^{b}. The strain will change
by $\Delta\varepsilon_{ij}$ to ε_{ij}^{b}. Then the postulate requires

$$\int_{\varepsilon_{ij}^{a}}^{\varepsilon_{ij}^{b}} (\sigma_{ij} - \sigma_{ij}^{a})d\varepsilon_{ij} > 0 \qquad\qquad (15)$$

The value of the integral is strongly path dependent in the plastic range,
but is of course independent of path for any elastic material.

Absolute Minimum Principles in Elasticity

The inequality (15) is a formal expression of the requirement
that the shaded triangles of Fig. 1 be on the positive strain energy side
of the stress-strain curve. Although for a non-linear elastic material
ε_{ij}^{b} depends upon ε_{ij}^{a} as well as on $\Delta\sigma_{ij}$, the integral is path independ-
ent. Choosing a straight line path in stress space from σ_{ij}^{a} to σ_{ij}^{b}
it is obvious that inequality (15) may be continued as

$$0 < \int_{\varepsilon_{ij}^{a}}^{\varepsilon_{ij}^{b}} (\sigma_{ij}-\sigma_{ij}^{a})d\varepsilon_{ij} < (\sigma_{ij}^{b}-\sigma_{ij}^{a})(\varepsilon_{ij}^{b}-\varepsilon_{ij}^{a}) \equiv \Delta\sigma_{ij}\Delta\varepsilon_{ij} \qquad (16)$$

Also, from

$$(\sigma_{ij}^{b}-\sigma_{ij}^{a})(\varepsilon_{ij}^{b}-\varepsilon_{ij}^{a}) = \int_{a}^{b} d[(\sigma_{ij}-\sigma_{ij}^{a})(\varepsilon_{ij}-\varepsilon_{ij}^{a})]$$

$$= \int_{\varepsilon_{ij}^{a}}^{\varepsilon_{ij}^{b}} (\sigma_{ij}-\sigma_{ij}^{a})d\varepsilon_{ij} + \int_{\sigma_{ij}^{a}}^{\sigma_{ij}^{b}} (\varepsilon_{ij}-\varepsilon_{ij}^{a})d\sigma_{ij} \qquad (17)$$

$$0 < \int_{\sigma_{ij}^{a}}^{\sigma_{ij}^{b}} (\varepsilon_{ij}-\varepsilon_{ij}^{a})d\sigma_{ij} < \Delta\sigma_{ij}\Delta\varepsilon_{ij} \qquad (18)$$

Inequality (18) expresses the requirement that the shaded triangles be as
shown in Fig. 2a, b, c and not as in Fig. 2d, e.

Materials of the type of Figs. 1d, e and 2d, e are thus excluded
from our consideration although not necessarily from physical reality. For

elastic materials which follow the postulated behavior, comparison of

(16) with (9) and of (18) with (14) proves that the potential energy and

the complementary energy of the true state each is an absolute minimum

$$
\begin{aligned}
P.E.^t &\leq P.E.^c \\
C.E^t &\leq C.E^E
\end{aligned}
\tag{19}
$$

the equality sign applying only to the trivial case of the admissible state

c or E coinciding with the true state.

Deformation or Total Theories of Plasticity

If no distinction is made between loading and unloading, or if

each point of the body is assumed to be at the maximum load intensity in

its history, deformation theories postulate a unique relation between stress

and total strain. Although physically unacceptable in general because

plastic deformation is path dependent and irreversible, such theories do

in some instances lead to very useful results. Under the assumptions

mentioned there is no need to consider deformation theory further as it is

essentially non-linear elasticity. No matter how elaborate the stress-

strain relation, if the material postulated is stable, the principles of

minimum potential and minimum complementary energy apply without any change.

If, on the other hand, loading is taken to be non-linear but

unloading is assumed to follow a linear elastic-relation, the inconsistency

of deformation theory becomes of primary importance. The mathematical

and physical meaning of solutions then becomes quite obscure.

Work-Hardening Relations Involving Increments of Stress and Strain

The fundamental postulate of positive work by an external agency

has very far reaching implications. As shown in Fig. 3 the plastic strain

increment or strain rate vector $\varepsilon_{ij}^{\prime p}$ must be normal to the yield or loading

surface at a smooth point and between normals to adjacent points at a
corner. At a smooth point

$$\varepsilon_{ij}' = C_{ijk\ell}\,\sigma_{k\ell}' + G\,\frac{\partial f}{\partial \sigma_{ij}}\,\frac{\partial f}{\partial \sigma_{k\ell}}\,\sigma_{k\ell}' \qquad (20)$$

where $C_{ijk\ell}\sigma_{k\ell}'$ is the elastic response, G and f are functions of the
state of the material which may include strain and the history of loading
as well as the existing state of stress. G may in addition be a homogeneous
function of order zero in the stress rate σ_{ij}'. In pictorial terms, Fig. 3,
G may depend upon the direction of σ_{ij}' but doubling σ_{ij}' doubles ε_{ij}'.
In all stress-strain relations in use today, G is taken as completely
independent of σ_{ij}' so that in the form

$$\varepsilon_{ij}'^{p} = H_{ijk\ell}\,\sigma_{k\ell}' \qquad (21)$$

the $H_{ijk\ell}$ likewise are independent of $\sigma_{k\ell}'$. The coefficients $H_{ijk\ell}$ which
appear similar to the $C_{ijk\ell}$ of linear anisotropic elasticity may be
horribly complicated functions of the present state and prior history.

If a corner is considered to be a set of intersecting loading
surfaces [7]-[10] each of which makes its independent contribution to
the plastic strain rate, Fig. 4, then

$$\varepsilon_{ij}' = C_{ijk\ell}\sigma_{k\ell}' + {}^{(1)}H_{ijk\ell}\sigma_{k\ell}' + {}^{(2)}H_{ijk\ell}\sigma_{k\ell}' + \text{---}{}^{(m)}H_{ijk\ell}\sigma_{k\ell}' + \text{----}$$
$$= (C_{ijk\ell} + B_{ijk\ell})\sigma_{k\ell}' \qquad (22)$$

It is important to keep in mind that the coefficients ${}^{(m)}H_{ijk\ell}$ are to be
taken as zero unless σ_{ij}' has an outward pointing normal component just as
in (20) where the plastic term must be chosen as zero if unloading takes
place. Stress rate vectors having different directions often will activate
different loading surfaces so that despite the apparent linearity of (22)

it is not true in general that the strain rate produced by two stress rates acting simultaneously is the sum of the strain rates for each individually. In fact, at a corner the combination of two stress rates each of which individually produces plastic action may result in an unloading, Fig. 5. Even when the $^{(m)}H_{ijk\ell}$ are independent of σ_{ij}', the $B_{ijk\ell}$ will be functions of the stress rate (order zero).

The basic postulate requires the initial rate of work by the external agency to be positive, therefore

$$\frac{1}{2}\sigma_{ij}'\epsilon_{ij}' = \frac{1}{2}(C_{ijk\ell} + B_{ijk\ell})\sigma_{ij}'\sigma_{k\ell}' > 0 \tag{23}$$

and because the elastic component is recoverable

$$\frac{1}{2}\sigma_{ij}'\epsilon_{ij}'^p = \frac{1}{2}B_{ijk\ell}\sigma_{ij}'\sigma_{k\ell}' > 0 \tag{24}$$

for a work-hardening material unless $\epsilon_{ij}'^p \equiv 0$.

In the demonstration of the uniqueness theorem for stress and strain rates [7]-[10] the entire point lies in the proof of

$$(^a\sigma_{ij}' - {}^b\sigma_{ij}')(^a\epsilon_{ij}' - {}^b\epsilon_{ij}') > 0, \quad a \neq b \tag{25}$$

where $\underline{a}$ and $\underline{b}$ are two assumed solutions for the rates from the stress point σ_{ij}^L. If an infinitesmal time, arbitrarily chosen as unity, is permitted to elapse the two stress states are $\sigma_{ij}^L + {}^a\sigma_{ij}'$ and $\sigma_{ij}^L + {}^b\sigma_{ij}'$, Fig. 6. At a smooth point of the loading surface it is possible to go from stress point b to stress point a, Fig. 6a, b, or from Stress point $\underline{a}$ to $\underline{b}$, Fig. 6a, c and change the strain by $^a\epsilon_{ij}' - {}^b\epsilon_{ij}'$ or $^b\epsilon_{ij}' - {}^a\epsilon_{ij}'$ respectively in accord with (20) and (23). The work postulate for the $\underline{b}$ to $\underline{a}$ case then gives, see (9)

$$\int_b^a (\sigma_{ij} - \sigma_{ij}^L - {}^b\sigma_{ij}')d\epsilon_{ij} \equiv \int_{b_{\epsilon_{ij}'}}^{a_{\epsilon_{ij}'}} (\sigma_{ij}' - {}^b\sigma_{ij}')d\epsilon_{ij}' > 0 \tag{26}$$

The result (25) therefore is established [7] but the value of the
integral itself is of interest here.

$$0 < \frac{1}{2}\,{}^a\sigma_{ij}^{\,\prime}\,{}^a\varepsilon_{ij}^{\,\prime} + \frac{1}{2}\,{}^b\sigma_{ij}^{\,\prime}\,{}^b\varepsilon_{ij}^{\,\prime} - {}^b\sigma_{ij}^{\,\prime}\,{}^a\varepsilon_{ij}^{\,\prime} < ({}^a\sigma_{ij}^{\,\prime} - {}^b\sigma_{ij}^{\,\prime})({}^a\varepsilon_{ij}^{\,\prime} - {}^b\varepsilon_{ij}^{\,\prime}) \quad (27)$$

so that as for (18)

$$0 < \int_{{}^b\sigma_{ij}^{\,\prime}}^{{}^a\sigma_{ij}^{\,\prime}} (\varepsilon_{ij}^{\,\prime} - {}^b\varepsilon_{ij}^{\,\prime})d\sigma_{ij}^{\,\prime} = \frac{1}{2}\,{}^a\sigma_{ij}^{\,\prime}\,{}^a\varepsilon_{ij}^{\,\prime} + \frac{1}{2}\,{}^b\sigma_{ij}^{\,\prime}\,{}^b\varepsilon_{ij}^{\,\prime} - {}^a\sigma_{ij}^{\,\prime}\,{}^b\varepsilon_{ij}^{\,\prime} \quad (28)$$

For the a to b path, (27) and (28) merely interchange and the result
is therefore unaffected.

When at a corner there are two or more loading surfaces, Fig. 6d,
the permissible path may be from b to a for one set of plastic strain
rates and from a to b for another. As both (27) and (28) apply to
each path, remembering to count the elastic strain rates but once, it is
clear that (27) and (28) apply just as they are even for this very complicated
case.

Two Minimum Theorems for Incremental Work-Hardening Theories

In general, theorems which hold in the elastic range cannot be
expected to apply in the plastic. As a consequence of irreversibility,
the uniqueness theorem for work-hardening theories of plasticity is in
terms of the increments or rates of stress and strain and not the stresses
and strains themselves. The equivalent variational or minimum principles
likewise will be in terms of rates. Following the procedure established
previously, the principle of virtual work is written for the rates

$$\int_V \sigma_{ij}^{\,\prime*}\varepsilon_{ij}^{\,\prime}dV - \int_{A_u} T_i^{\,\prime*}u_i^{\,\prime}dA - \int_{A_T} T_i^{\,\prime*}u_i^{\,\prime}dA - \int_V F_i^{\,\prime*}u_i^{\,\prime}dV = 0 \quad (29)$$

Equation (29) applies to the actual rates as it does for any compatible strain rate distribution and any stress rate field in equilibrium.

When the true stress rates are varied while satisfying equilibrium and the boundary conditions on A_T ($T_i' = 0$), the analog to (10) is

$$\int_V \Delta \sigma_{ij}' \epsilon_{ij}'^t \, dV - \int_{A_u} \Delta T_i' u_i' dA = 0 \tag{30}$$

The analog to complementary energy density (11) is simply

$$\omega(\sigma_{ij}') = \frac{1}{2} \sigma_{ij}' \epsilon_{ij}'(\sigma_{ij}') \tag{31}$$

corresponding to linear elasticity because the stress-strain relations are time independent as exhibited by (21).

The form suggested for a variational principle similar to (12) is

$$\delta_{\sigma'} \left[\int_V \omega(\sigma_{ij}'^t) dV - \int_{A_u} T_i'^t u_i' dA \right] = 0 \tag{32}$$

This complementary rate principle will be valid whenever the $H_{ijk\ell}$ of (21) are independent of stress rate. All currently used forms (20) for smooth loading surfaces are in this category. At a corner, however, as previously explained the $H_{ijk\ell}$ of (22) which are non-zero depend upon the direction of σ_{ij}'. Therefore, $\delta_{\sigma'} H_{ijk\ell} \neq 0$ for some or all directions of loading and (32) is not valid.

A complementary rate minimum principle in a form equivalent to (19) would be much more valuable

$$\int_V \omega(\sigma_{ij}'^t) dV - \int_{A_u} T_i'^t u_i' dA \leq \int_V \omega(\sigma_{ij}'^E) dV - \int_{A_u} T_i'^E u_i' dA \tag{33}$$

The right hand side is algebraically larger than the left by

$$\int_V [\omega(\sigma_{ij}'^E) - \omega(\sigma_{ij}'^t)]dV - \int_{A_u} (T_i'^E - T_i'^t)u_i' dA$$

$$= \int_V [\frac{1}{2}(\sigma_{ij}'^E \epsilon_{ij}'^E - \sigma_{ij}'^t \epsilon_{ij}'^t) - (\sigma_{ij}'^E - \sigma_{ij}'^t)\epsilon_{ij}'^t] dV \tag{34}$$

where $\epsilon_{ij}'^E$ is computed from $\sigma_{ij}'^E$, Equation (22).

The integrand of (34) may be rewritten as

$$\frac{1}{2}\sigma_{ij}'^E \epsilon_{ij}'^E + \frac{1}{2}\sigma_{ij}'^t \epsilon_{ij}'^t - \sigma_{ij}'^E \epsilon_{ij}'^t \tag{35}$$

Comparison with (28) shows that the fundamental work postulate requires (35) to be positive and thus guarantees the minimum principle (33) although (32) does not apply.

If now the strain rates are varied in the virtual work expression for the true rate state while satisfying $u' = 0$ on A_u, the equivalent of (4) is

$$\int_V \sigma_{ij}'^t \Delta\epsilon_{ij}' dV - \int_{A_T} T_i' \Delta u_i' dA - \int_V F_i' \Delta u_i' dV = 0 \tag{36}$$

The analog to strain energy density (5) would be

$$w(\epsilon_{ij}') = \frac{1}{2}[\sigma_{ij}'(\epsilon_{ij}')]\epsilon_{ij}' \tag{37}$$

if (22) can be inverted as

$$\sigma_{ij}' = A_{ijk\ell}\,\epsilon_{k\ell}' \tag{38}$$

Such inversion is not possible if the material is incompressible in either the elastic or the elastic-plastic range. This difficulty can be circumvented for such materials by solving for the stress rate deviation

$$s_{ij}' = a_{ijk\ell}\,\epsilon_{k\ell}' \tag{39}$$

and writing

$$w(\varepsilon_{ij}') = \frac{1}{2} s_{ij}' \varepsilon_{ij}' = \frac{1}{2} \sigma_{ij}' \varepsilon_{ij}' \tag{40}$$

It can be shown, then, that a minimum potential rate principle holds

$$\int_V w(\varepsilon_{ij}'^t) dV - \int_{A_T} T_i' u_i'^t dA - \int_V F_i' u_i'^t dA \leq \int w(\varepsilon_{ij}'^c) dV$$
$$- \int_{A_T} T_i' u_i'^c dA - \int_V F_i' u_i'^c dA \tag{41}$$

The right hand side exceeds the left by

$$\int_V [w(\varepsilon_{ij}'^c) - w(\varepsilon_{ij}'^t)] dV - \int_V \sigma_{ij}'^t (\varepsilon_{ij}'^c - \varepsilon_{ij}'^t) dV \tag{42}$$

when the surface traction rate and body force rate integrals are replaced
by volume integrals of stress rates. The sum of the integrands is

$$\frac{1}{2} \sigma_{ij}'^c \varepsilon_{ij}'^c + \frac{1}{2} \sigma_{ij}'^t \varepsilon_{ij}'^t - \sigma_{ij}'^t \varepsilon_{ij}'^c \tag{43}$$

Comparison of (43) with (27) proves (41) and once again
demonstrates that the minimum principles follow from the fundamental
postulate or definition of work-hardening.

Incrementally non-linear stress-strain relations have not been
studied in any detail. If G in (20) or $H_{ijk\ell}$ in (21) depend on the
direction of σ_{ij}', the usual proof of the uniqueness theorem breaks down.
On the other hand, if uniqueness is assured the basic postulate in the
form of (26) or the first inequality (28) will ensure the validity of the
absolute minimum principles (33) and (41) corresponding to complementary
and potential energy. A simple assumption which leads to uniqueness is
that the plastic strain rate is a monotonically increasing function of the
normal component of σ_{ij}'. Although apparently a very reasonable postulate,

as a consequence of the proportionality between rates of stress and strain
for a given direction of σ_{ij}' it is equivalent, unfortunately, to a
reduction to linearity.

Restricted Minimum Theorems

The complete parallelism of the developments of the minimum
principles for the elastic and for the elastic-plastic cases may have blurred
the basic approach. There are but a few independent combinations possible
for stress, stress rate, strain, and strain rate and it is worth while running
through some of them. As uniqueness of rates only is guaranteed it is not
likely that any new theorems of true generality will result for work-hardening
materials.

Suppose a theorem is desired for the plastic range which contains
the stresses and the strains themselves. Equations (2), (3), (4), and (10)
are written exactly as for elastic bodies. Again (5) and (11) would be
suggested by the form of (4) and (10). Now, however, the path of loading
is important and it is not true in general that $\int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = W$ is a
function of final strain only, nor is $\int_0^{\sigma_{ij}} \varepsilon_{ij} d\sigma_{ij}$ a function of stress
only. Nevertheless, if at each point of the material there has been no
unloading from any of the loading surfaces the irreversibility is not
apparent to the material. In this very limited sense, minimum complementary
and potential energy theorems hold for a very restricted and yet possibly
useful class of alternative admissible states [11].

Next suppose that a theorem is desired for strain rates and
stress. Virtual work is then written in the form

$$\int_V \sigma_{ij} \varepsilon_{ij}' dV - \int_{A_u} T_i u_i' dA - \int_{A_T} T_i u_i' dA - \int_V F_i u_i' dV = 0 \tag{44}$$

Varying the strain rate system from the true state without changing the

displacement rate boundary conditions leads to

$$\int_V \sigma_{ij}^{\,t} \Delta\varepsilon_{ij}' dV - \int_{A_T} T_i \Delta u_i' dA - \int_V F_i \Delta u_i' dV = 0 \qquad (45)$$

The variational principle suggested is

$$\delta_{\varepsilon'} \left[\int_V \varphi(\varepsilon_{ij}'^{\,t}) dV - \int_{A_T} T_i u_i'^{\,t} dA - \int_V F_i u_i'^{\,t} dV \right] = 0 \qquad (46)$$

where

$$\frac{\partial \varphi^t}{\partial \varepsilon_{ij}'^{\,t}} = \sigma_{ij}^{\,t} \qquad (47)$$

Such a principle can have meaning only if ε_{ij}' determines σ_{ij} or at least $\sigma_{ij}\varepsilon_{ij}'$. As this will not be true in general, (46) can be valid for a restricted set of loading paths or special materials at most.

Varying the equilibrium system in the familiar manner results in

$$\delta_\sigma \left[\int_V \Psi(\sigma_{ij}^{\,t}) dV - \int_{A_u} T_i^t u_i' dA \right] = 0 \qquad (48)$$

which has meaning if and only if σ_{ij} determines ε_{ij} or at least $\sigma_{ij}'\varepsilon_{ij}'$

$$\frac{\partial \Psi^t}{\partial \sigma_{ij}} = \varepsilon_{ij}'^{\,t} \qquad (49)$$

Therefore, (48) cannot apply to a work-hardening material.

Another possible set of theorems relates stress rates and strain. The corresponding equations then are found by interchanging primes and no primes in (44)-(49) and the end results are

$$\delta_\varepsilon \left[\int_V \Phi(\varepsilon_{ij}^{\,t}) dV - \int_{A_T} T_i' u_i^t dA - \int_V F_i' u_i^t dV \right] = 0 \qquad (50)$$

where and only if ϵ_{ij} determines σ_{ij}'

$$\frac{\partial \Phi^t}{\partial \epsilon_{ij}^{\ t}} = \sigma_{ij}'^{\ t} \tag{51}$$

and

$$\delta_{\sigma'} \left[\int_V \Psi(\sigma_{ij}'^{\ t}) dV - \int_{A_u} T_i'^{\ t} u_i dA \right] = 0 \tag{52}$$

where and only if σ_{ij}' determines ϵ_{ij}

$$\frac{\partial \Psi}{\partial \sigma_{ij}'^{\ t}} = \epsilon_{ij}^{\ t} \tag{53}$$

Such theorems as these are therefore inappropriate for work-hardening
materials as defined. They will have limited validity at least for
materials whose state is described by surfaces in strain space rather
than in stress space [12][13].

Elastic-Perfectly Plastic Material

A perfectly plastic material may be defined directly or equally
well be considered as the limiting case of a work-hardening one for which
all subsequent loading surfaces coincide with the initial yield surface,
f = k, bounding purely elastic action. Unlimited plastic deformation may
occur at yield. As no stress increment is required for flow at yield, the
work done by an external agency may be zero when plastic deformation takes
place. The plastic strain rate vector is normal to the yield surface in
the extended sense, Fig. 3.

The stress-strain relation at a smooth point on the yield
surface is

$$\epsilon_{ij}' = \epsilon_{ij}'^{\ e} + \epsilon_{ij}'^{\ p} = C_{ijk\ell} \sigma_{k\ell}' + \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{54}$$

and at a corner

$$\varepsilon_{ij}' = C_{ijk\ell}\sigma_{k\ell}' + \overset{1}{\lambda}\frac{\partial f_1}{\partial\sigma_{ij}} + \overset{2}{\lambda}\frac{\partial f_2}{\partial\sigma_{ij}} + ----\tag{55}$$

where each (fixed) intersecting yield surface makes its own contribution. The λ are homogeneous of degree minus one in time because the stress-strain relations are independent of time. Each λ is to be taken as zero unless the stress point is on its yield surface and remains there in the interval under consideration. They are otherwise indeterminate for a homogeneous state of stress.

As a consequence of a fixed yield surface and normality,

$$\sigma_{ij}'\varepsilon_{ij}'^{p} = 0\tag{56}$$

for all permissible σ_{ij}'. Therefore

$$2\omega_p(\sigma_{ij}') = \sigma_{ij}'\varepsilon_{ij}' = \sigma_{ij}'\varepsilon_{ij}'^{e} = C_{ijk\ell}\,\sigma_{ij}'\sigma_{k\ell}' > 0\tag{57}$$

Also, the total strain rate vector is resolved uniquely into an elastic and a plastic component at each stress point on the yield surface. At an interior point, the strain rate is, of course, purely elastic. Therefore, when the existing stress is known and the strain rate is given the stress rate is determined except, again, for an incompressible material in which the stress deviation rate is determined [similar to (38) and (39)]

$$2w_p(\varepsilon_{ij}') = \sigma_{ij}'\varepsilon_{ij}' = \sigma_{ij}'\varepsilon_{ij}'^{e} = A_{ijk\ell}\,\varepsilon_{ij}'^{e}\varepsilon_{k\ell}'^{e} > 0\tag{58}$$

All the equations (26)-(43) are valid, therefore, with ω_p replacing ω and w_p replacing w. The discussion of what happens at a corner is simplified for a perfectly plastic material. If the strain

rate vector points outward from the yield surface and lies between the normals to adjacent points, see Fig. 3, it is purely plastic and the stress rate is identically zero. If the strain rate vector points outward but has a component tangent to the adjacent surface, that portion of the surface governs and the stress point is then effectively at a smooth point on the yield surface.

It is not surprising that a minimum complementary rate principle and a minimum potential rate principle applies. As has been stated, a perfectly plastic material is a limiting case of a work hardening one as all successive loading surfaces approach the initial yield surface. The lack of limitation on $\varepsilon_{ij}'^{p}$ for a homogeneous state of stress does not matter because, at a given stress point, $\sigma_{ij}'\varepsilon_{ij}'^{p}$ is zero when the strain rate and the stress rate are related and zero or negative when they represent two independent states.

Additional theorems of some generality would be expected for a perfectly plastic material because the yield surface does not depend upon loading. The plastic strain rate vector is normal to the surface and so determines the stress point itself or at worst a straight line or plane of the surface, Fig. 7. Actually $\varepsilon_{ij}'^{p}$ determines the rate of dissipation, $\sigma_{ij}\varepsilon_{ij}'^{p}$, uniquely. The total strain rate vector by itself does not provide any such information for an elastic-plastic material. However, for a plastic-rigid material or for an elastic-plastic material at the limit load [14], the elastic strain rates are identically zero. The total strain rates are then plastic only and do determine the dissipation and the stress to a considerable extent.

Under the restriction of zero elastic strain rate, following steps (44)-(47), ε_{ij}' does determine $\sigma_{ij}\varepsilon_{ij}' = \varphi$ and the principle (46) is established. Following steps (48) and (49), $\sigma_{ij}'\varepsilon_{ij}'$ is now zero so

that $Y = 0$ and (48) applies.

Absolute minimum principles can be established which are generalizations of the Markov [15] and Hill [16] principles and are in fact equivalent to the limit theorems [14]. The minimum principle for (46) is

$$\int_V \varphi(\epsilon_{ij}'^t)dV - \int_{A_T} T_i u_i'^t dA - \int_V F_i u_i'^t dV \leq \int_V \varphi(\epsilon_{ij}'^c)dV - \int_{A_T} T_i u_i'^c dA - \int_V F_i u_i'^c dV$$

$$(59)$$

where $\epsilon_{ij}'^c$, $u_i'^c$ is any compatible system taken as plastic only, and $\varphi = [\sigma_{ij}(\epsilon_{ij}')]\epsilon_{ij}'$ is the dissipation function. The right hand side is algebraically greater than the left by the volume integral of

$$\varphi(\epsilon_{ij}'^c) - \varphi(\epsilon_{ij}'^t) - \sigma_{ij}^t(\epsilon_{ij}'^c - \epsilon_{ij}'^t) = \sigma_{ij}^c \epsilon_{ij}'^c - \sigma_{ij}^t \epsilon_{ij}'^c = (\sigma_{ij}^c - \sigma_{ij}^t)\epsilon_{ij}'^c \quad (60)$$

which is positive or zero in accordance with the basic work postulate [4] or equally well from the convexity of the yield surface and the normality of the plastic strain rate vector which themselves are consequences of the postulate.

The upper bound limit theorem [14] may be obtained from (59) by observing that if u_i' vanishes on A_u, the left hand side is zero from virtual work and

$$\int_A T_i u_i'^c dA + \int_V F_i u_i'^c dV \leq \int_V \varphi(\epsilon_{ij}'^c)dV \quad (61)$$

The minimum principle corresponding to (48) may be written as a maximum principle by multiplying through by minus one

$$\int_{A_u} T_i^t u_i' dA \geq \int_{A_u} T_i^E u_i' dA \quad (62)$$

Proof follows directly from virtual work as the left hand side exceeds the right by the volume integral of

$$(\sigma_{ij}^{t} - \sigma_{ij}^{E})\epsilon_{ij}'^{t} \qquad (63)$$

which is positive or zero just as (60). It is (63) which is the key in the lower bound theorem of limit analysis [14]

The yield value and the yield criterion as well need not be the same at each point of the material as nowhere in the proofs do such restrictions appear. Therefore (59) and (62) apply equally well to a rigid-work hardening material at each stage of loading.

As mentioned in the previous section, theorems involving total strain are not appropriate for the type of materials postulated in the paper [13] so that the list of simple minimum principles seems exhausted.

Conclusion

A systematic procedure is presented for establishing variational and minimum principles. The virtual work expression is written in terms of the quantities for which a theorem is sought and a variation is tried which suggests a possible principle. Use is then made of a basic postulate for stable materials without time effects which had been formulated previously [4]: in the very strictest sense work cannot be extracted from the stressed material and the system of forces acting upon it. Substitution of the relation between the quantities which is given directly by the fundamental postulate provides immediate proof of the valid absolute minimum principles.

Minimum potential energy and minimum complementary energy theorems (19) are established for linear and non-linear elastic bodies and for deformation theories of plasticity by virtual work (2), the appropriate

variation (4) or (10), and inequalities (16) or (18) as given by the
postulate.

Corresponding theorems [2][5] in which rates replace total
quantities (41) and (33) are established for work-hardening and for
perfectly plastic materials by virtual work (29), the appropriate variation
(36) or (30), and inequalities (27) or (28) given by the postulate. All
incrementally linear and the most complicated combinations of incrementally
linear forms, Fig. 4, are included.

Extended theorems (59) and (62) involving stress quantities and
strain and displacement rates [15][16] are established for rigid-perfectly
plastic materials or elastic-perfectly plastic at limit loading or collapse.
The steps are virtual work (44), the variations (45) or (48), and inequalities
given by the postulate as indicated following (60) or (63).

Bibliography

[1] W. Prager and J. L. Synge, "Approximations in Elasticity Based on the
       Concept of Function Space", Quarterly of Applied Mathematics, v. 5
       (1947) pp. 241-269.

     J. B. Diaz and H. J. Greenberg, "Upper and Lower Bounds for the Solution
       of the First Boundary Value Problem of Elasticity", Quarterly of
       Applied Mathematics, v. 6 (1948) pp. 326-331.

[2] H. J. Greenberg, "On the Variational Principles of Plasticity", ONR
       Report A11-S4, Division of Applied Mathematics, Brown University
       (March 1949).

[3] E. Reissner, "On a Variational Theorem in Elasticity", Journal of
       Mathematics and Physics, v. 29, n. 2 (July 1950) pp. 90-95.

     K. Washizu, "On the Variational Principles of Elasticity and Plasticity",
       ONR Report 2S-18, Aeroelastic and Structures Research Laboratory,
       Massachusetts Institute of Technology (March 1955).

[4] D. C. Drucker, "Some Implications of Work-Hardening and Ideal Plasticity",
       Quarterly of Applied Mathematics, v. 7, n. 4 (January 1950) pp. 411-
       418.

     D. C. Drucker, "A More Fundamental Approach to Plastic Stress-Strain
       Relations", Proceedings First U.S. National Congress of Applied
       Mechanics, A.S.M.E. (1951) pp. 487-491.

[5] R. Hill, "The Mathematical Theory of Plasticity", Clarendon Press,
       Oxford (1950).

[6] W. Prager and P. G. Hodge, Jr., "Theory of Perfectly Plastic Solids",
       John Wiley & Sons (1951).

[7] D. C. Drucker, "On Uniqueness in the Theory of Plasticity", Quarterly
       of Applied Mathematics, v. 14, n. 1 (April 1956) pp. 35-42.

[8] W. Koiter, "Stress-Strain Relations, Uniqueness and Variational Theorems
       for Elastic-Plastic Materials with a Singular Yield Surface",
       Quarterly of Applied Mathematics, v. 11 (1953) pp. 350-353.

[9] J. L. Sanders, "Plastic Stress-Strain Relations Based on Infinitely
       Many Plane Loading Surfaces", Proceedings 2nd U.S. National
       Congress of Applied Mechanics, A.S.M.E. (1954) pp. 455-460.

[10] B. Budiansky, "Fundamental Theorems and Consequences of the Slip Theory
       of Plasticity", Ph.D. Thesis, Brown University, Providence, R. I.
       (1950).

[11] P. G. Hodge, Jr., "Minimum Principles of Piecewise Linear Isotropic
       Plasticity", ONR Report 298, Polytechnic Institute of Brooklyn,
       (August 1955).

[12] D. Trifan, "A Minimum Principle of Plasticity", Quarterly of Applied
         Mathematics, v. 13 (1955) pp. 337-339.

[13] W. Prager, "On Limiting States of Deformation", ONR Report C11-9
         Division of Applied Mathematics, Brown University (February 1956).

[14] D. C. Drucker, W. Prager, H. J. Greenberg, "Extended Limit Design
         Theorems for Continuous Media", Quarterly of Applied Mathematics,
         v. 9, n. 4 (January 1952) pp. 381-389.

[15] A. A. Markov, "On Variational Principles in the Theory of Plasticity",
         Prikladnaia Matematika i Mekhanika, v. 11 (1947) pp. 339-350.

[16] R. Hill, "A Variational Principle of Maximum Plastic Work in Classical
         Plasticity", Quarterly Journal of Mechanics and Applied Mathematics,
         v. 1 (1948) pp. 18-28.
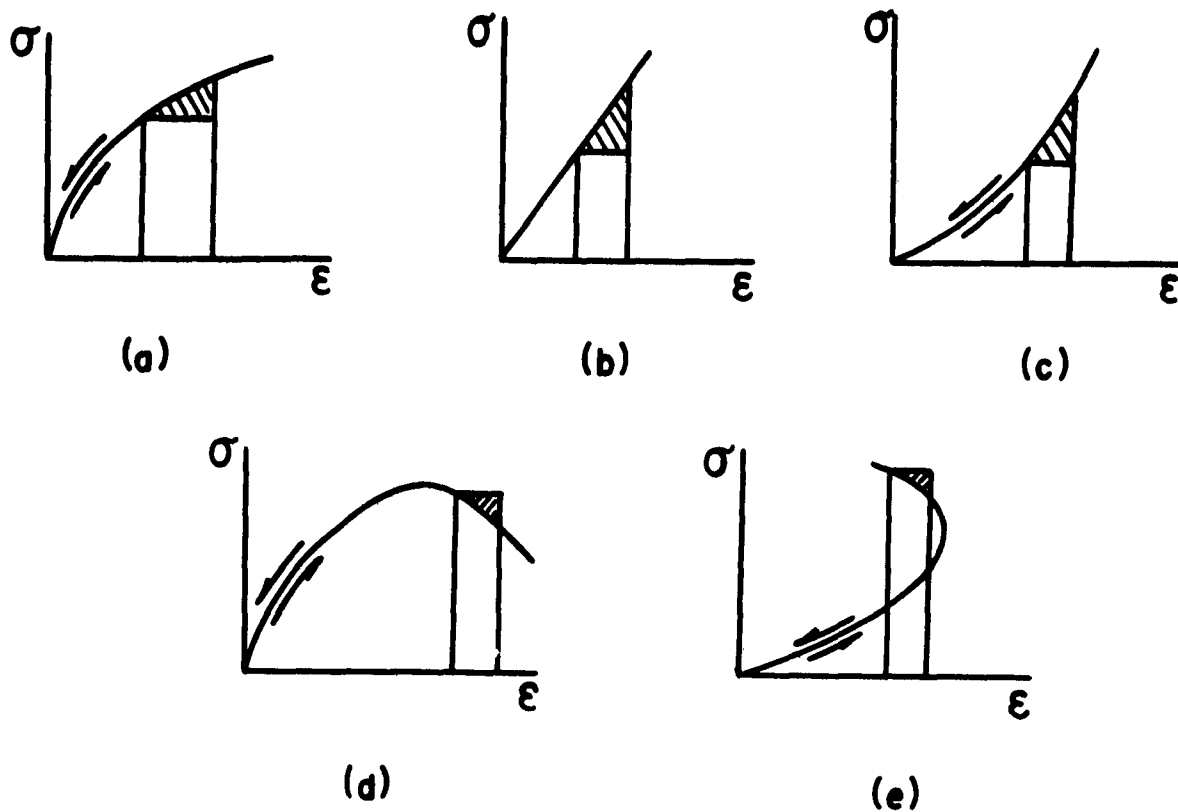
FIG.I POTENTIAL ENERGY IS A MINIMUM FOR A STABLE
ELASTIC MATERIAL (a), (b), (c)
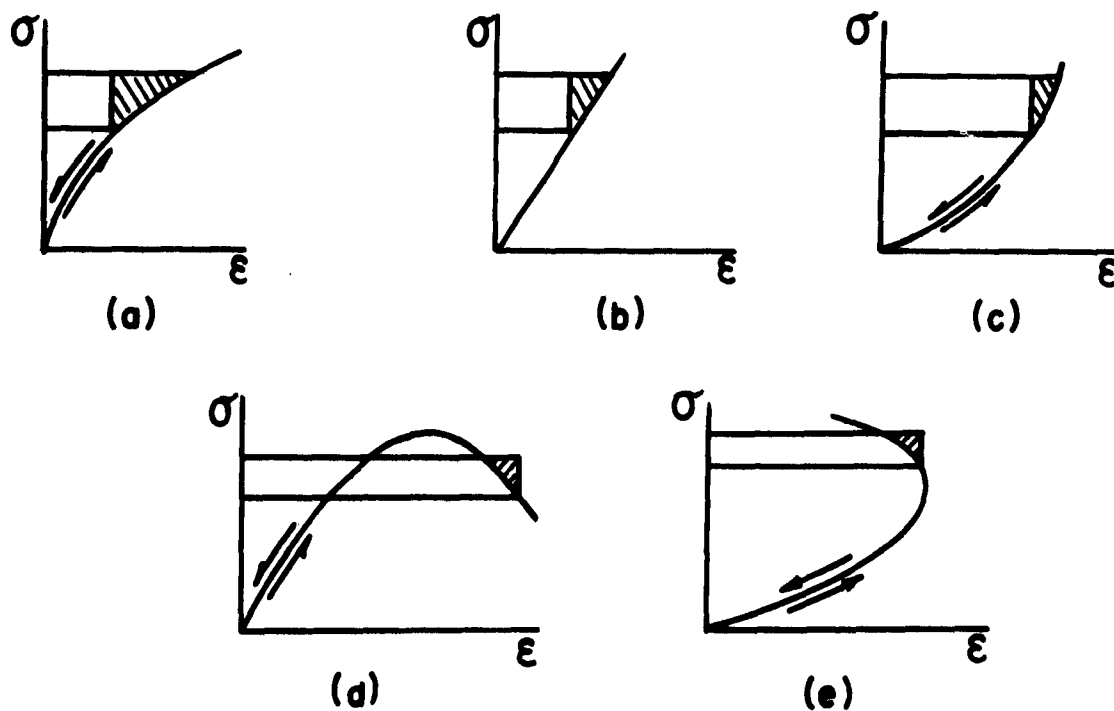
FIG. 2 COMPLEMENTARY ENERGY IS A MINIMUM FOR A STABLE
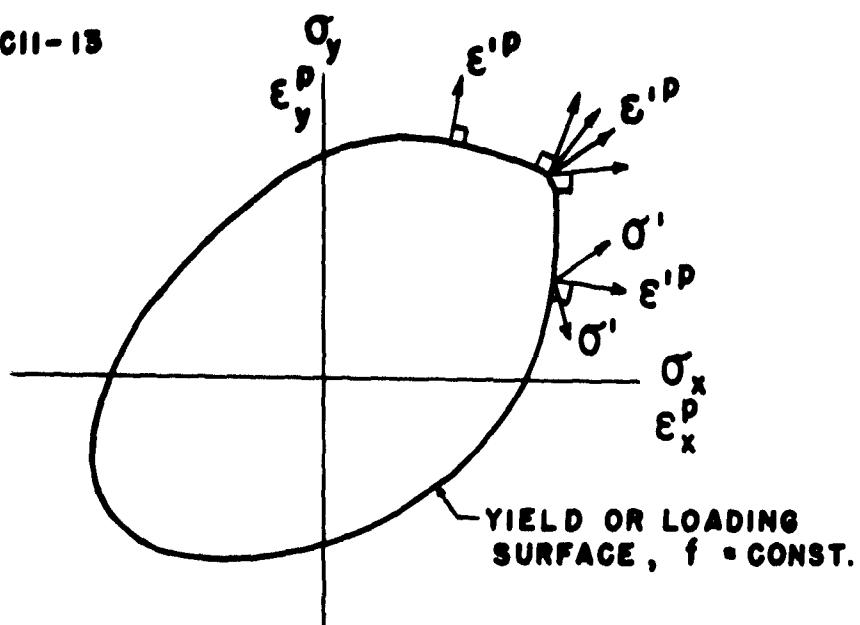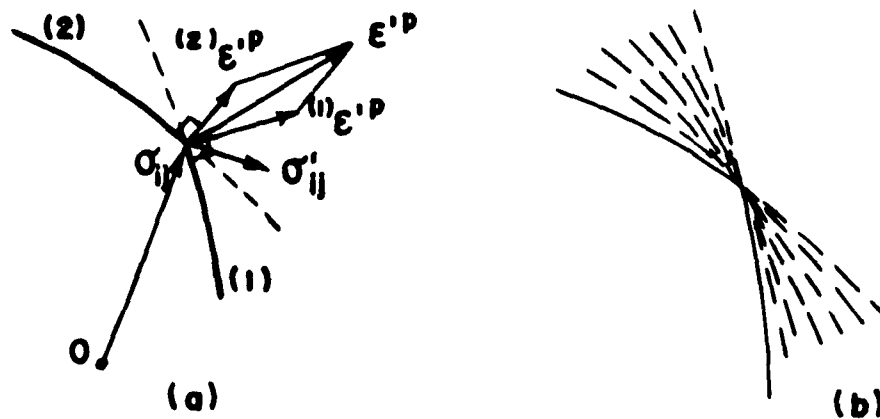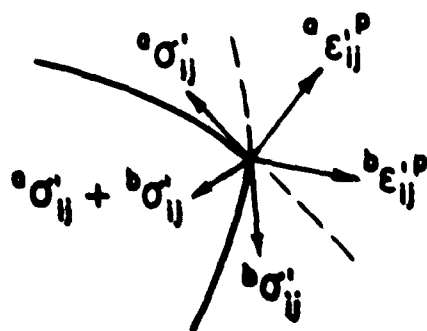ELASTIC MATERIAL (a), (b), (c)

FIG. 3   NORMALITY OF PLASTIC STRAIN INCREMENT (RATE)

FIG. 4   A CORNER AS AN   INTERSECTION OF TWO OR MORE

LOADING   SURFACES

Two only are drawn for convenience of illustration in 4a

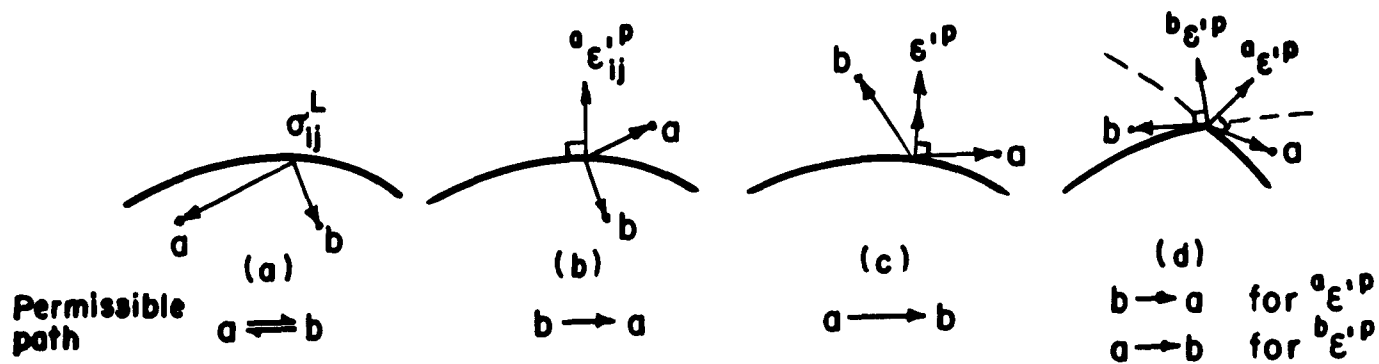but there may be infinitely many as symbolized in 4b
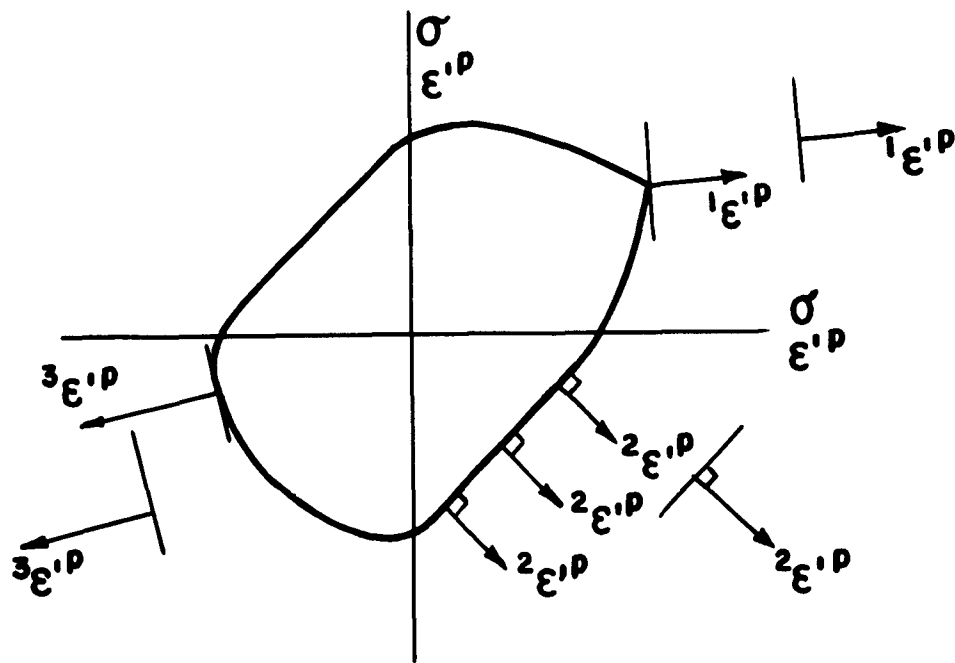
FIG. 6 PERMISSIBLE PATHS

FIG. 7 THE PLASTIC STRAIN RATE VECTOR DETERMINES
THE RATE OF DISSIPATION OF ENERGY, $\sigma_{ij}\, \epsilon_{ij}^{'p}$,
AND THE STRESS POINT σ_{ij} ITSELF EXCEPT AT
A FLAT SPOT